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Permuting tri-(f, g)-derivations on lattices

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Abstract. In this paper as a generalization of permuting tri-derivations and permuting tri-f-derivations of a lattice, we introduce the notion of Permuting tri-(f, g)-derivations of a lattice. If the function g is equal to the function f then the permuting tri-(f, g)-derivation is the permuting tri-f-derivation defined in [19]. Also if we choose the functions f and g the identity functions both then the derivation we define coincides with the derivation defined in [15].

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1. Introduction

The lattice algebra has an important role and has many applications in information theory, information retrieval, information access controls and cryptanalysis. For more information one can study [1, 2, 4, 8, 9, 11, 12, 16, 20].

Szász introduced the notion of lattice derivation and gave interesting results [17]. Also in [10] the author studied this lattice derivation. In [18] Xin et al. improved derivation for a lattice and discussed some related properties. They gave some equivalent conditions under which a derivation is isotone for lattices with a greatest element, modular lattices and distributive lattices.

In [6] Çeven and Öztürk gave a generalization of derivation on a lattice which was defined in [18]. Çeven in [5] introduced the symmetric bi derivations on lattices. Çeven and Öztürk [7] discussed some properties of symmetric bi-(σ, τ)-derivations in near-rings. The author investigated some related properties. He characterized the distributive and modular lattices by the trace of symmetric bi derivations. Ozbal and Firat in [13] introduced the notion of symmetric f-bi-derivation of a lattice. They characterized the distributive lattice by symmetric f-bi-derivation.

In [14] Öztürk introduced the notion of permuting tri-derivations in rings and proved some results. Also in [15] Öztürk et al. introduced the permuting tri-derivations in lattices. Yazarli and Öztürk generalized the permuting tri-derivations to permuting tri-f-derivations in [19]. In this paper as a generalization of [15] and [19] we introduce the notion of permuting tri-(f, g)-derivations of a lattice. We give illustrative example. We define the isotone permuting tri-(f, g)-derivation and get some interesting results about isotoneness. We characterize the distributive and isotone lattices by permuting tri-(f, g)-derivations.

2. Preliminaries

Definition 2.1 ([3]). Let L be a nonempty set endowed with operations $\wedge$ and $\vee$. If $(L, \wedge, \vee)$ satisfies the following conditions for all $x, y, z \in L$

(1) $x \wedge x = x$, $x \vee x = x$
(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$
(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$

then L is called a lattice.

Definition 2.2 ([3]). A lattice L is distributive if the identity (5) or (6) holds.

(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(6) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Definition 2.3 ([3]). Let $(L, \wedge, \vee)$ be a lattice. A binary relation $\leq$ is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Lemma 2.4 ([18]). *Let $(L, \wedge, \vee)$ be a lattice. Define the binary relation $\leq$ as the Definition 2.3. Then $(L, \leq)$ is a poset and for any $x, y \in L$, $x \wedge y$ is the g.l.b. of $\{x, y\}$ and $x \vee y$ is the l.u.b. of $\{x, y\}$.*

Definition 2.5 ([15]). Let L be a lattice. A mapping $D : L \times L \times L \to L$ is called permuting if it satisfies the following conditions $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) = D(z, x, y) = D(z, y, x)$ for all $x, y, z \in L$.

Definition 2.6 ([15]). A mapping $d : L \to L$ defined by $d(x) = D(x, x, x)$ is called the trace of D where D is a permuting mapping.

Definition 2.7 ([15]). Let L be a lattice and D be a permuting tri-derivation on L. We call D *joinitive* if it satisfies

$$D(x \vee w, y, z) = D(x, y, z) \vee D(w, y, z)$$

for all $x, y, z, w \in L$.

Definition 2.8 ([19]). Let L be a lattice. A permuting mapping $D : L \times L \times L \to L$ is called permuting tri-f-derivation if there exists a function $f : L \to L$ such that

$$D(x \wedge w, y, z) = (D(x, y, z) \wedge f(w)) \vee (f(x) \wedge D(w, y, z))$$

for all $x, y, z, w \in L$.

3. Permuting tri-(f,g)-derivations of lattices

Definition 3.1. Let L be a lattice and $D : L \times L \times L \to L$ be a permuting mapping. D is called permuting tri-(f,g)-derivation of L if there exist functions $f, g : L \to L$ such that

$$D(x \wedge w, y, z) = (D(x,y,z) \wedge f(w)) \vee (g(x) \wedge D(w,y,z)) \tag{3.1}$$

for all $x, y, z, w \in L$.

Obviously a permuting tri-(f,g)-derivation D on L satisfies the relations

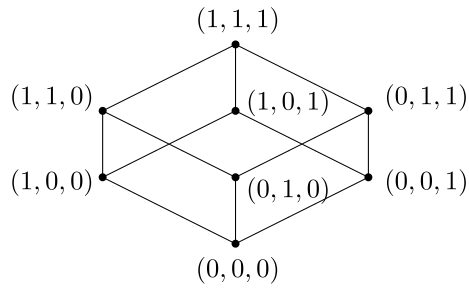
$$D(x, y \wedge w, z) = (D(x,y,z) \wedge f(w)) \vee (g(y) \wedge D(x,w,z))$$

and

$$D(x, y, z \wedge w) = (D(x,y,z) \wedge f(w)) \vee (g(z) \wedge D(x,y,w)).$$

For special case, if we get the function g equal to the function f then our derivation coincides with the permuting tri-f-derivation in [19]. Also if we get the functions f and g the identity functions then our derivation is the derivation defined in [15].

Example 3.2. Let us take the lattice $\mathbf{1^3}$ which is given by the following diagram.



Define a function D on L by

$$D(x,y,z) = \begin{cases} 1, & (x,y,z) = (0,0,0) \\ 0, & (x,y,z) = (0,0,1), (0,1,0), (0,1,1), (1,0,0) \\ 0, & (x,y,z) = (1,0,1), (1,1,0), (1,1,1) \end{cases}$$

then D is not a permuting tri-derivation of L since

$$\begin{aligned} 1 &= D(0,0,0) \\ &= D(0 \wedge 0, 0, 0) \\ &\neq (D(0,0,0) \wedge 0) \vee (0 \wedge D(0,0,0)) \\ &= 0. \end{aligned}$$

If we define functions f and g on L respectively $f(0) = f(1) = 1$ and $g(0) = 0$, $g(1) = 1$, then $f \neq g$ and D defined above is a permuting tri-(f,g)-derivation of L. Also if we get $f(0) = g(0) = 1$ and $f(1) = g(1) = 1$ then $f = g$ and D is a permuting tri-f-derivation of L.

Proposition 3.3. *Let L be a lattice and d be the trace of permuting tri-(f,g)-derivation D on L. Then*
$$d(x) \leq (f(x) \vee g(x))$$
for all $x \in L$.

Proof. Since $x \wedge x = x$ for all $x \in L$ and from the definition of trace we have
$$d(x) = D(x,x,x) = D(x \wedge x, x, x) = (D(x,x,x) \wedge f(x)) \vee (g(x) \wedge D(x,x,x).$$
Since $D(x,x,x) \wedge f(x) \leq f(x)$ and $D(x,x,x) \wedge g(x) \leq g(x)$, we get that $d(x) \leq f(x) \vee g(x)$. □

Proposition 3.4. *Let L be a lattice and D be a permuting tri-(f,g)-derivation on L. Then $D(x,y,z) \leq f(x) \vee g(x)$, $D(x,y,z) \leq f(y) \vee g(y)$ and $D(x,y,z) \leq f(z) \vee g(z)$ for all $x,y,z \in L$.*

Proof. Since $x \wedge x = x$ for all $x \in L$ then we have
$$D(x,y,z) = D(x \wedge x, y, z) = (D(x,y,z) \wedge f(x)) \vee (g(x) \wedge D(x,y,z).$$
Since $D(x,y,z) \wedge f(x) \leq f(x)$ and $D(x,y,z) \wedge g(x) \leq g(x)$, we get
$$D(x,y,z) \leq f(x) \vee g(x).$$
Similarly $D(x,y,z) \leq f(y) \vee g(y)$ and $D(x,y,z) \leq f(z) \vee g(z)$. □

Proposition 3.5. *Let D be a permuting tri-(f,g)-derivation on a lattice L. If L has a least element 0, such that $f(0) = 0$ and $g(0) = 0$, then $D(0,y,z) = 0$.*

Proof. From Proposition 3.3 we have $D(x,y,z) \leq (f(x) \vee g(x))$ for all $x,y,z \in L$. Since 0 is the least element of the lattice then $0 \leq D(0,y,z) \leq (f(0) \vee g(0)) = 0$. Then we say that $D(0,y,z) = 0$. □

Proposition 3.6. *Let L be a lattice with a greatest element 1 and D be a permuting tri-(f,g)-derivation on L such that $f(1) = g(1) = 1$. Then the following are valid:*

(i) *If $f(x) \leq D(1,y,z)$ and $g(x) \leq D(1,y,z)$ then $D(x,y,z) = (f(x) \vee g(x))$.*
(ii) *If $f(x) \geq D(1,y,z)$ and $g(x) \geq D(1,y,z)$ then $D(x,y,z) \geq D(1,y,z)$.*

Proof. (i) Since
$$\begin{aligned} D(x,y,z) &= D(x \wedge 1, y, z) \\ &= (D(x,y,z) \wedge f(1)) \vee (g(x) \wedge D(1,y,z)) \\ &= D(x,y,z) \vee g(x) \end{aligned}$$
then we have
$$g(x) \leq D(x,y,z). \tag{3.2}$$
Similarly since $x \wedge 1 = 1 \wedge x$ then we can write
$$\begin{aligned} D(x,y,z) &= D(1 \wedge x, y, z) \\ &= (D(1,y,z) \wedge f(x)) \vee (g(1) \wedge D(x,y,z)) \\ &= f(x) \vee D(x,y,z). \end{aligned}$$
Again we have
$$f(x) \leq D(x,y,z). \tag{3.3}$$

From (3.2) and (3.3) we have

$$(f(x) \vee g(x)) \leq D(x, y, z).$$

From Proposition 3.3 we have $D(x, y, z) \leq (f(x) \vee g(x))$. Finally we have

$$(f(x) \vee g(x)) \leq D(x, y, z) \leq (f(x) \vee g(x)),$$

which completes the proof.

(ii) Since

$$\begin{aligned} D(x, y, z) &= D(x \wedge 1, y, z) \\ &= (D(x, y, z) \wedge f(1)) \vee (g(x) \wedge D(1, y, z)) \\ &= D(x, y, z) \vee D(1, y, z) \end{aligned}$$

then we have

$$D(x, y, z) \geq D(1, y, z).$$

□

Theorem 3.7. *Let L be a distributive lattice and D be a permuting tri-(f, g)-derivation on L with the trace d. Then*

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (g(x) \wedge d(y)) \vee \{(g(x) \wedge f(y)) \wedge [D(x, x, y) \vee D(x, y, y)]\}$$

for all $x, y \in L$.

Proof. From the definition of the trace we have

$$\begin{aligned} & d(x \wedge y) = D(x \wedge y, x \wedge y, x \wedge y) \\ & = (D(x, x \wedge y, x \wedge y) \wedge f(y)) \vee (g(x) \wedge D(y, x \wedge y, x \wedge y)) \\ & = \{[(D(x, x, x \wedge y) \wedge f(y)) \vee (g(x) \wedge D(x, y, x \wedge y))] \wedge f(y)\} \\ & \quad \vee \{g(x) \wedge [(D(y, x, x \wedge y) \wedge f(y)) \vee (g(x) \wedge D(y, y, x \wedge y))]\} \\ & = \{(d(x) \wedge f(y)) \vee (g(x) \wedge D(x, x, y) \wedge f(y)) \vee (g(x) \wedge D(x, y, y) \wedge f(y))\} \\ & \quad \vee \{(g(x) \wedge d(y)) \vee (g(x) \wedge D(y, x, x) \wedge f(y)) \vee (g(x) \wedge D(y, x, y) \wedge f(y))\} \\ & = (d(x) \wedge f(y)) \vee (g(x) \wedge d(y)) \vee \{(g(x) \wedge f(y)) \wedge ((D(x, y, y) \vee D(x, x, y))\}. \end{aligned}$$

This completes the proof. □

Corollary 3.8. *Let L be a distributive lattice and D be a permuting tri-(f, g)-derivation on L with the trace d. Then,*

(i) $(g(x) \wedge f(y)) \wedge D(x, x, y) \leq d(x \wedge y)$, $(g(x) \wedge f(y)) \wedge D(x, y, y) \leq d(x \wedge y)$.
(ii) $g(x) \wedge d(y) \leq d(x \wedge y)$.
(iii) $d(x) \wedge f(y) \leq d(x \wedge y)$.

Proof. (i), (ii) and (iii) are easily seen from the theorem above. □

Corollary 3.9. *Let L be a distributive lattice and 1 be the greatest element of L. For special cases $(g(x) \wedge f(1)) \wedge D(x, x, 1) \leq d(x \wedge 1) = d(x)$ and $g(x) \wedge d(1) \leq d(x \wedge 1) = d(x)$ for all $x \in L$.*

Corollary 3.10. *Let L be a distributive lattice, D be a permuting tri-(f, g)-derivation on L with the trace d and 1 be the greatest element of L. Then*

(i) $g(x) \geq d(1) \Rightarrow d(x) \geq d(1)$.

(ii) *$g(x) \leq d(1)$ and $f(x) \leq g(x)$ for all $x \in L \Rightarrow g(x) = d(x)$.*

Definition 3.11. Let L be a lattice and D be a permuting tri-(f,g)-derivation on L with the trace d.

(i) If $x \leq y$ implies $d(x) \leq d(y)$ then d is called an trace isotone mapping.
(ii) If d is one to one, d is called a trace monomorphic mapping.
(iii) If d is onto then d is called an trace epic mapping.

Definition 3.12. Let L be a lattice and D be a permuting tri-(f,g)-derivation on L, if $x \leq y$ implies $D(x,w,z) \leq D(y,w,z)$ then D is called an isotone permuting tri-(f,g)-derivation on L

Proposition 3.13. *Let L be a lattice and d be the trace of permuting tri-(f,g)-derivation D on L Then the following conditions are equivalent;*

(i) *d is an isotone mapping.*
(ii) *$dx \vee dy \leq d(x \vee y)$.*

Proof. (1)$\Rightarrow$(2) Suppose that d is an isotone mapping. We know that $x \leq x \vee y$ and $y \leq x \vee y$. Since d is isotone then $d(x) \leq d(x \vee y)$ and $d(y) \leq d(x \vee y)$. Hence we get $d(x) \vee d(y) \leq d(x \vee y)$.

(2)$\Rightarrow$(1) Suppose that $d(x) \vee d(y) \leq d(x \vee y)$ and $x \leq y$. Then we get $d(x) \leq d(x) \vee d(y) \leq d(x \vee y) = d(y)$. This means that d is an isotone mapping. ☐

Theorem 3.14. *Let L be a lattice with greatest element 1 and D be an isotone permuting tri-(f,g)-derivation on L. Let $f(1) = g(1) = 1$ and either $f(x) \geq g(x)$ or $f(x) \leq g(x)$ for all $x \in L$. Then*

$$D(x,y,z) = (f(x) \vee g(x)) \wedge D(1,y,z)$$

for all $x, y, z \in L$.

Proof. Suppose that D is an isotone permuting tri-(f,g)-derivation on L. Then $D(x,y,z) \leq D(1,y,z)$ for all $x,y,z \in L$. Now suppose that $f(x) \geq g(x)$ for $x \in L$. Then we have $D(x,y,z) \leq f(x) \vee g(x) = f(x)$. From this we get $D(x,y,z) \leq f(x) \wedge D(1,y,z)$. Also

$$\begin{aligned}
D(x,y,z) &= D((x \vee 1) \wedge x, y, z) \\
&= [(D(x \vee 1), y, z) \wedge f(x)] \vee [g(x \vee 1) \wedge D(x,y,z)] \\
&= [D(1,y,z) \wedge f(x)] \vee [g(1) \wedge D(x,y,z)] \\
&= [D(1,y,z) \wedge f(x)] \vee [1 \wedge D(x,y,z)] \\
&= [D(1,y,z) \wedge f(x)] \vee D(x,y,z) \\
&= D(1,y,z) \wedge f(x)
\end{aligned}$$

Since $f(x) \vee g(x) = f(x)$ then we get

$$D(x,y,z) = (f(x) \vee g(x)) \wedge D(1,y,z).$$

Now suppose that $f(x) \leq g(x)$ for $x \in L$. Then similarly we have $D(x, y, z) \leq f(x) \vee g(x) = g(x)$. From this we get $D(x, y, z) \leq g(x) \wedge D(1, y, z)$. Also

$$\begin{aligned} D(x,y,z) &= D(x \wedge (x \vee 1), y, z) \\ &= [(D(x,y,z) \wedge f(x \vee 1)] \vee [g(x) \wedge D((x \vee 1), y, z)] \\ &= [D(x,y,z) \wedge f(1)] \vee [g(x) \wedge D(1,y,z)] \\ &= [D(x,y,z) \wedge 1] \vee [g(x) \wedge D(1,y,z)] \\ &= D(x,y,z) \vee [g(x) \wedge D(1,y,z)] \\ &= g(x) \wedge D(1,y,z) \end{aligned}$$

Since $f(x) \vee g(x) = g(x)$ then we get

$$D(x,y,z) = (f(x) \vee g(x)) \wedge D(1,y,z).$$

This completes the proof. □

4. Conclusion

In this paper as a generalization of permuting tri-derivation and permuting tri-f-derivation of a lattice we introduced the notion of permuting tri-(f,g)-derivation of a lattice. We defined the isotone permuting tri-(f,g)-derivation and got some interesting results about isotoneness. We characterized the distributive and isotone lattices by permuting tri-(f,g)-derivation. If the function g is equal to the function f then the permuting tri-(f,g)-derivation is the permuting tri-(f,g)-derivation defined in [19]. Also if we choose the functions f and g the identity functions both, then the derivation we define coincides with the derivation defined in [15].

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